Non-global jet evolution at finite N_c

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Resummations of soft gluon emissions play an important role in many applications of QCD, among them jet observables and small x saturation effects. Banfi, Marchesini, and Smye have derived an evolution equation for non-global jet observables that exhibits a remarkable analogy with the BK equation used in the small x context. Here, this analogy is used to generalize the former beyond the leading N_c approximation. The result shows striking analogy with the JIMWLK equation describing the small x evolution of the color glass condensate. A Langevin description allows numerical implementation and provides clues for the formulation of closed forms for amplitudes at finite N_c. The proof of the new equation is based on these amplitudes with ordered soft emission. It is fully independent of the derivation of the JIMWLK equation and thus sheds new light also on this topic.

1 Introduction

The starting point for the issues discussed and developed in this paper is a beautiful exposition by Banfi, Marchesini, and Smye (BMS) [1] of an evolution equation describing interjet observables in hard QCD processes in which one asks for the energy flow $E_{\rm out}$ into a region away from all hard jets. With $E_{\rm out}$ much smaller than the hard scale of the jet, QCD radiation into this region is small and these observables offer an ideal opportunity to study nonperturbative effects. The central result of [1] is an evolution equation that resums, for large $Q/E_{\rm out}$ and *in the large N_c limit*, all leading terms arising from large angle soft emission. In the following, I will refer to this equation as the BMS equation. Subsequent analysis of the consequences implied by this equation presented in the same publication confirm features found using numerical methods by Dasgupta and Salam [2, 3]. The equation itself is nonlinear in the part describing contributions arising *inside* the jet regions and linear outside. The contributions inside, quite surprisingly, have a striking analogy with the Balitsky-Kovchegov (BK) equation [4, 5, 6], which resums small x corrections in the nonlinear domain. There it is used to describe saturation phenomena that occur in DIS at small x, possibly off large hadronic targets. For small angle emission in a certain frame, the analogy becomes even more striking: the contributions of BMS inside the jet regions become identical to the BK equation. That such an analogy, if it exists at all, is restricted to the in-region, is not surprising: in the derivation of the BK equation one assumes, in fact requires, the presence of non-linear effects *everywhere* and any geometric exclusiveness of the type imposed by jets was simply never considered. What appeared to be more surprising is the relation of jet physics to BFKL dynamics [7, 8], which dominates the BK equation in the small density limit. This link has been established in [9] and indeed comes into play in the limit of small angle emission in a judiciously chosen frame.

I will be concerned with a different question: the generalization of the BMS equation to finite N_c based on the fact that such is already available on the BK side of the above analogy.

The BK equation is the limit of a more general equation, the JIMWLK (Jalilian-Marian Iancu McLerran Weigert Leonidov Kovner) equation [10, 11, 12, 13, 14, 15, 16, 17, 18][1], a functional equation equivalent to an infinite hierarchy of coupled evolution equations known as the Balitsky hierarchy [4, 5].

BK emerges from JIMWLK by means of a factorization assumption strongly linked with the large N_c limit. As both BK and BMS equations use eikonalized soft emission techniques, both in a strongly ordered setting, it is only natural that there will be a host of technical similarities in both situations that may serve as a

[1] The order of names was chosen by Al Mueller to give rise to the acronym JIMWLK, pronounced "gym walk".

vehicle to derive a consistent generalization of the BMS equation that goes beyond the large N_c limit. This derivation will be the goal of this paper.

In Sec. 2 I will shortly present both the BMS and BK equations and highlight their structural analogy. This will provide me with a tentative identification of the transition probabilities used in the BMS equation with averages of "dipole type" operators akin to those appearing in the BK equation, albeit in a different kinematic domain and geometrical setting.

Using this identification in Sec. 3, I will suggest a nonlinear evolution equation that generalizes the BMS equation in the same way as JIMWLK generalizes the BK equation. From a single equation for a single function one generalizes to a functional equation, equivalent to an infinite hierarchy of equations for an infinite tower of correlators.

I will identify real-emission and virtual correction parts of the resulting functional Fokker-Planck equation and discuss the issues of real-virtual cancellation and highlight the fact that this is in direct correspondence with infrared finiteness.

Sec. 4 will provide the translation of the resulting functional Fokker-Planck equation to an equivalent Langevin description.This will allow to identify explicitly both real and virtual contributions to the amplitudes in a closed form. At the same time one gains a tool to study this evolution equation at finite N_c numerically, in analogy to what has been done already in the JIMWLK case [19]. It will also provide insights into the structures that will be needed to derive the underlying Fokker-Planck equation.

The remainder of the paper is devoted to a derivation of this equation from first principles. Due to the functional nature of the equation, the most economical way to proceed is to consider generating functionals for the objects of interest. This will allow me to write closed functional expressions for all the relevant amplitudes and transition probabilities at *finite* N_c in terms of a gluon emission operator whose form is motivated by the comparison with JIMWLK – the amplitudes found are of course the relevant QCD amplitudes of [20, 21].

In Sec. 5 I will define generating functionals for amplitudes that describe the transition of any bare jet configuration with an arbitrary but fixed number of hard particles that emit any number of soft gluons $g_1 \dots g_n$ in the strongly ordered domain $\omega_1 \gg \dots \gg \omega_n$, all softer than the original set of hard particles. At this stage I will focus on real emission. My initial example will be based on a bare $q\bar{q}$ configuration.

Sec. 6 is used to formulate generating functionals for transition probabilities corresponding to the above amplitudes that will then serve as the objects for which the evolution equation is derived.

This is done in Sec. 7 by considering derivatives of the expressions for real emission with respect to the phase space boundary in complete analogy to the method of BMS. Virtual corrections are then included in the usual way with a coefficient fixed to ensure real-virtual cancellation in the infrared. The result is a proof of the equation suggested in Sec. 2 and a verification of the fact that it indeed resums *all* finite N_c corrections in the kinematic domain under consideration.

Conclusions and a short discussion on possible cross fertilizations between the two fields will conclude this paper in Sec. 8.

2 The analogy of BMS and BK equations

To present the analogy, let me give a short account of both equations, starting with BMS. I will closely follow the exposition in [1].

The observable considered by BMS is given by

$$\Sigma_{e^+e^-}(E_{\text{out}}) = \sum_n \int \frac{d\sigma_n}{\sigma_T} \cdot \Theta\left(E_{\text{out}} - \sum_{h \in C_{\text{out}}} \omega_h\right) . \tag{1}$$

Here E_{out} is the sum of all energies of soft gluons emitted into the region outside some predefined jet region, $\mathcal{C}_{\text{out}}$ and $d\sigma_n$ the n-hadron distribution and σ_T the total cross section. The dependence on jet geometry and total energy Q is understood. The process is dominated by iterative soft emission from eikonalized gluons that

are not deflected from their original trajectory. The natural kinematic variables are thus the directions of the emitters ($a, b, k, p, \bar{p}$ in what follows) and the corresponding energies. Phase space integrals correspondingly will involve integrals over solid angles and energies.

A Mellin transformation represents the cross section in terms of transition probabilities $G_{ab}(E, E_{\rm out})$

$$\Sigma_{e^+e^-}(E_{\rm out}) \;=\; \int \frac{d\nu\, e^{\nu E_{\rm out}}}{2\pi i\nu}\, G_{p\bar{p}}(E,\nu^{-1}) \simeq G_{p\bar{p}}(E, E_{\rm out})\,, \tag{2}$$

were a saddle point approximation has been taken.

$G_{ab}(E, E_{\rm out})$, or $G_{ab}(E)$ where the explicit $E_{\rm out}$ dependence is not needed, satisfies the BMS equation

$$E\partial_E\, G_{ab}(E) = \int \frac{d^2\Omega_k}{4\pi}\, \bar{\alpha}_s w_{ab}(k)\, [u(k)\, G_{ak}(E)\cdot G_{kb}(E) - G_{ab}(E)] \;, \tag{3}$$

or equivalently

$$\begin{aligned} E\partial_E\, G_{ab}(E) = &-E\partial_E\, R^{(0)}_{ab}(E)\cdot G_{ab}(E) \\ &+ \int \frac{d^2\Omega_k}{4\pi}\, \bar{\alpha}_s w_{ab}(k)\, u(k)\, [\, G_{ak}(E)\cdot G_{kb}(E) - G_{ab}(E)\,] \;. \end{aligned} \tag{4}$$

$\bar{\alpha}_s := \alpha_s N_c/\pi$, $R^{(0)}_{ab}(E)$ is called the single log Sudakov radiator for bremsstrahlung emission

$$R^{(0)}_{ab}(E) \;=\; \int_0^E \frac{d\omega}{\omega} \int \frac{d^2\Omega_k}{4\pi}\, \bar{\alpha}_s\, w_{ab}(k)\, [1-u(k)] \;=\; \Delta\cdot r_{ab}\,, \tag{5}$$

and

$$w_{ab}(k) = \frac{(p_a p_b)}{(p_a k)(k p_b)} = \frac{1-\cos\theta_{ab}}{(1-\cos\theta_{ak})(1-\cos\theta_{kb})} \tag{6}$$

is the soft emission kernel. Conventions here are adapted to lightlike momenta and the energies, ω_p, ω_q have been factored out in (pq) compared to the 4-vector product: $p.q = \omega_p\omega_q(pq) = \omega_p\omega_q(1-\cos\theta_{pq})$.

The jet geometry is encoded in the definition of $u(k)$, which emerges from a Mellin factorization of the energy constraint in Eq. (1):

$$\Theta\Big(E_{\rm out} - \sum_{i\in\mathcal{C}_{\rm out}} \omega_i\Big) = \int \frac{d\nu\, e^{\nu E_{\rm out}}}{2\pi i\nu} \prod_i u(k_i)\,, \quad u(k) = \Theta_{\rm in}(k) + e^{-\nu\omega}\Theta_{\rm out}(k)\,, \tag{7}$$

with the Θ functions having support inside and outside the jet regions respectively. In fact, BMS show that to good accuracy one may take $u(k)$ to restrict the phase space integrals to the outside region for the Sudakov radiator term and to the inside region for the remainder of Eq. (4).

For more details on the ingredients as well as the physics of this equation see [1, 2, 3, 20, 21] and references therein.

At this point I want to highlight the strategy employed by BMS in the derivation of Eq. (3). This evolution equation was derived from the knowledge of the structure of the real emission part alone. The contribution of the virtual corrections was included after the fact using the requirement of real-virtual cancellation in the infrared.

The real emission part of G needed for this argument can be written as

$$G^{(\rm real)}_{p\bar{p}}(E, E_{\rm out}) = 1 + \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} \left\{ \bar{\alpha}_s \frac{d\omega_i}{\omega_i} \frac{d^2\Omega_i}{4\pi} u(k_i)\, \Theta(E-\omega_i) \right\} W_n(p k_1 \ldots k_n \bar{p}) \tag{8}$$

where the phase space of soft gluons is cut by E and W_n is the large N_c factorized version of the transition probability of a color singlet into $(q\bar{q})_{\rm hard} g^n_{\rm soft}$.

$$W_n(p k_1 \ldots k_n \bar{p}) \;=\; \frac{(p\bar{p})}{(pk_1)(k_1k_2)\ldots(k_n\bar{p})} \;. \tag{9}$$

The line of argument then starts by taking a logarithmic derivative $E\partial_E$ of Eq. (8) which uniquely fixes the quadratic term in Eq. (3). This term is affected by the jet-geometry as signalled by the factor $u(k)$. The virtual corrections are then taken into account by subtraction of a linear term with a coefficient that ensures infrared finiteness through real virtual cancellation. Virtual corrections occur globally, both inside and outside the jet region, thus the factor $u(k)$ is absent. This yields the second term in Eq. (3).

That this indeed ensures infrared finiteness is best understood after splitting off the Sudakov term as in Eq. (4). Noting that the integration in the Sudakov term is restricted to the out region one recognizes that there is no danger of encountering any ill effects from the poles in the kernel in this term. For the other term one needs to prove (see [1]) that $G_{aa}(E) = 1$. Then the quadratic and the linear terms cancel where the kernels diverge, rendering the expression finite.

The BK equation on the other hand involves small x kinematics in which in the dipole approximation is taken. The latter corresponds to the large N_c limit of the above. While again one deals with hard leading particles that emit softer ones in an eikonal manner, they all share a common longitudinal direction, selected by the geometry of the collision event. Instead of directions away from the hard creation vertex of the jet(s), one is left with (2d) coordinates in the plane transverse to the collision axis that characterize the location of the hard emitters. To set the coordinate system in the small x case, I follow the common choice to represent the hard degrees of freedom as given by eikonal lines along the x^- direction (the direction of the probe). The target, which is separated from the probe by a large rapidity interval $\tau = \ln(1/x)$ (to leading accuracy it moves along the x^+ direction) then supplies the field these eikonal lines interact with. This plays a role similar to the soft emissions in the BMS case. Any x^+ dependence is frozen out by time dilation. The field variables then are given as

$$U_{\boldsymbol{x}} := P \exp\Big\{ -ig \int_{-\infty}^{\infty} dz^- A^+_{\rm soft}(x^+ = 0, \boldsymbol{x}, z^-) \Big\} \tag{10}$$

where $A^+_{\rm soft}(x^+ = 0, \boldsymbol{x}, z^-)$ is the soft/Weizäcker Williams field of the target. The basic object entering the DIS cross section is the dipole cross section, the impact parameter integral of

$$N_{\tau;\boldsymbol{xy}} = \langle \hat{N}_{\boldsymbol{xy}} \rangle_\tau \qquad\qquad \hat{N}_{\boldsymbol{xy}} = {\rm tr}(1 - U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}})/N_c \ . \tag{11}$$

$N_{\tau;\boldsymbol{xy}}$ is the dipole function, $\hat{N}_{\boldsymbol{xy}}$ the dipole operator and the average $\langle \ldots \rangle_\tau$ is to be thought to arise from an average of the U-configurations characterizing the scattering with a given target at a given τ. The τ dependence is target independent and IR safe and hence perturbatively calculable in form of an RG equation known as the BK equation. As usual, the nonperturbative information is relegated to its initial condition. To expose the similarity to the BMS equation it is useful to write the BK equation in terms of

$$S_{\tau;\boldsymbol{xy}} := \langle \hat{S}_{\boldsymbol{xy}} \rangle_\tau \qquad\qquad \hat{S}_{\boldsymbol{xy}} = {\rm tr}(U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{y}})/N_c \tag{12}$$

instead of $N_{\tau;\boldsymbol{xy}}$. Then it takes on the form

$$\partial_\tau S_{\tau;\boldsymbol{xy}} = \frac{\alpha_s N_c}{2\pi^2} \int d^2z \ \tilde{\mathcal{K}}_{\boldsymbol{xzy}} \left(S_{\tau;\boldsymbol{xz}} S_{\tau;\boldsymbol{zy}} - S_{\tau;\boldsymbol{xy}} \right) \tag{13}$$

where

$$\tilde{\mathcal{K}}_{\boldsymbol{xzy}} := \frac{(\boldsymbol{x} - \boldsymbol{y})^2}{(\boldsymbol{x} - \boldsymbol{z})^2 (\boldsymbol{z} - \boldsymbol{y})^2} \tag{14}$$

is the BK-kernel.

Ignoring for the moment the Sudakov radiator term in Eq. (4), the similarity of Eqns. (4) and (13) is striking. A one to one relationship of structures emerges if one maps directions onto transverse coordinates, the kernels onto each other and tries to view $G_{ab}(E)$ as the average of some ${\rm tr}(U^\dagger_a U_b)/N_c$ (where a and b represent the directions of the leading hard particles):

$$G_{ab}(E) \ \stackrel{?}{\longleftrightarrow} \ \langle {\rm tr}(U^\dagger_a U_b)/N_c \rangle_E \ . \tag{15}$$

Note that this is fully in line with the requirement that $G_{aa}(E) = 1$ needed to ensure real virtual cancellation in the second term of Eq. (4). While this appears to be quite intuitive and fully in line with the physical interpretation it is not clear how precisely to perform that latter part of the translation. One would ask, for instance, how in detail to arrive at a consistent definition of these Us and the averaging process suggested by Eq. (15) in the light of the tree like branching process underlying the physics of the BMS equation. Additional questions would be how to reconcile their role in probabilities with that in amplitudes and many more in the same vein.

Despite these open questions, the analogy is too strong to ignore. That the interpretation proposed in Eq. (15) should be possible suggests itself even more strongly, if one uses the observation made by BMS that in the small angle emission limit in a carefully chosen frame where the measure turns flat and the kernels agree completely:

$$w_{ab}(k) \to \tilde{\mathcal{K}}_{\hat{a}\hat{k}\hat{b}} \; . \tag{16}$$

This is the region where Marchesini and Mueller [9] have established the link between jet and BFKL-dynamics.

The benefit of demonstrating such a correspondence should be obvious to any reader familiar with the color glass condensate (CGC): If there is such an interpretation of the objects entering the BMS equation, then there is hope one might generalize it along similar lines as the BK equation, which can be viewed as the factorized limit of a more general functional equation, the JIMWLK equation which I will turn to next, in order to show how it reduces to BK under certain assumptions. This will be the analogy used to suggest a generalization of the BMS equation.

3 From JIMWLK to BK: generalizing BMS by analogy

To write the JIMWLK equation, consider the averaging procedure entering the definition of $S_{\tau;\boldsymbol{xy}}$ to be performed using a weight functional for eikonal lines U:

$$\langle \ldots \rangle_\tau = \int \hat{D}[U] \ldots \hat{Z}_\tau[U] \tag{17}$$

where $\hat{D}[U]$ denotes a functional Haar measure, in keeping with the group valued nature of the field variables U.

The JIMWLK equation describes τ dependence of the distribution functional $\hat{Z}_\tau[U]$. It was first written as a functional Fokker-Planck with a very compact evolution operator in [16] and reads

$$\partial_\tau \hat{Z}_\tau[U] = -H_{\text{FP}} \hat{Z}_\tau[U] \; . \tag{18}$$

The Fokker-Planck Hamiltonian is defined as

$$H_{\text{FP}} := \frac{1}{2} \; i\nabla^a_{\boldsymbol{x}} \chi^{ab}_{\boldsymbol{xy}} i\nabla^b_{\boldsymbol{y}} \tag{19a}$$

$$\chi^{ab}_{\boldsymbol{xy}} := -\frac{\alpha_s}{\pi^2} \int d^2z \; \mathcal{K}_{\boldsymbol{xzy}} \left[(1 - U^\dagger_x U_z)(1 - U^\dagger_z U_y) \right]^{ab} \tag{19b}$$

and can be shown to be positive definite. In Eqns. (19), both a summation and an integration convention is applied to repeated indices and coordinates. Where

$$\mathcal{K}_{\boldsymbol{xzy}} = \frac{(\boldsymbol{x} - \boldsymbol{z}) \cdot (\boldsymbol{z} - \boldsymbol{y})}{(\boldsymbol{x} - \boldsymbol{z})^2 (\boldsymbol{z} - \boldsymbol{y})^2} \tag{20}$$

is the JIMWLK kernel and $i\nabla^a_{\boldsymbol{x}}$ is a functional generalization of the left invariant vector field on the group manifold [There is *no* coordinate derivative in the above]. Its precise definition will be given shortly. Let me first provide two structural comments:

- Despite the apparent poles of the kernel at $\boldsymbol{z} = \boldsymbol{x}, \boldsymbol{y}$, the Hamiltonian will lead to explicitly finite evolution equations because of the structure of the U-dependence which vanishes at the potentially singular point. This is also the underlying reason for the cancellation of potential divergences in the BK limit.

- The equation is an equation for a distribution functional and thus a very compact way to represent an infinite number of (coupled) equations for individual correlators known as the Balitsky hierarchy [4]. To recover members of this hierarchy, say the evolution equation for a correlator $O[U]$, one simply multiplies both sides of (18) with $O[U]$ and the averages with (17). The result then is an evolution equation for $\langle O[U]\rangle_\tau$
$$\partial_\tau \langle O[U]\rangle_\tau = -\langle H_{\rm FP} O[U]\rangle_\tau \ . \tag{21}$$
Because of the nonlinear nature of $H_{\rm FP}$ the r.h.s. will not be expressible by $\langle O[U]\rangle_\tau$ alone but instead involves other correlators as well. The evolution equation of these new quantities will then also be needed, and the argument repeats itself, ultimetely leading to an infinite coupled hierarchy.

Returning to the left invariant vector fields I note that for present purposes they may be taken to be defined as a variational derivative according to

$$i\nabla^a_{\boldsymbol{x}} := -[U_{\boldsymbol{x}} t^a]\frac{\delta}{\delta U_{\boldsymbol{x},ij}} \ . \tag{22}$$

$\frac{\delta}{\delta U_{\boldsymbol{x},ij}}$ is the ordinary functional (or variational) derivative w.r.t. the components of the U field:

$$\frac{\delta}{\delta U_{\boldsymbol{x},ij}} U_{\boldsymbol{y},kl} = \delta_{ik}\delta_{jl}\delta^{(2)}_{\boldsymbol{xy}} \tag{23}$$

where $\delta^{(2)}_{\boldsymbol{xy}} := \delta^{(2)}(\boldsymbol{x}-\boldsymbol{y})$ for compactness. Operationally this then leads to

$$i\nabla^a_{\boldsymbol{x}} U_{\boldsymbol{y}} := -U_{\boldsymbol{x}} t^a \delta^{(2)}_{\boldsymbol{xy}} \ , \qquad i\nabla^a_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}} := t^a U^\dagger_{\boldsymbol{x}} \delta^{(2)}_{\boldsymbol{xy}} \ . \tag{24a}$$

There is, of course a corresponding definition for the right invariant vector fields $i\bar\nabla^a_{\boldsymbol{x}}$:

$$i\bar\nabla^a_{\boldsymbol{x}} U_{\boldsymbol{y}} := t^a U_{\boldsymbol{x}} \delta^{(2)}_{\boldsymbol{xy}} \ , \qquad i\bar\nabla^a_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}} := -U^\dagger_{\boldsymbol{x}} t^a \delta^{(2)}_{\boldsymbol{xy}} \ . \tag{24b}$$

Their main properties are the commutation relations (which I display leaving the functional nature aside for a second)

$$[i\nabla^a, i\nabla^b] = if^{abc} i\nabla^c \qquad [i\bar\nabla^a, i\bar\nabla^b] = if^{abc} i\bar\nabla^c \qquad [i\bar\nabla^a, i\nabla^b] = 0 \ . \tag{25a}$$

These then generate right and left translations respectively:[2]

$$e^{-i\omega^a (i\nabla^a)} U = U e^{i\omega^a t^a} \qquad e^{-i\omega^a (i\bar\nabla^a)} U = e^{-i\omega^a t^a} U \ . \tag{25b}$$

Functional forms of this of course involve an integral in the exponent:

$$e^{-i\int_x \omega^a_x (i\nabla^a_x)} U_y = U_y e^{i\omega^a_y t^a} \qquad e^{-i\int_x \omega^a_x (i\bar\nabla^a_x)} U_y = e^{-i\omega^a_y t^a} U_y \ . \tag{25c}$$

∇ and $\bar\nabla$ are interrelated by

$$i\nabla^a_{\boldsymbol{x}} = -[\tilde U^\dagger_{\boldsymbol{x}}]^{ab} i\bar\nabla^b_{\boldsymbol{x}}; \qquad i\bar\nabla^a_{\boldsymbol{x}} = -[\tilde U^\dagger]^{ab}_{\boldsymbol{x}} i\nabla^b_{\boldsymbol{x}} \tag{25d}$$

and "representation conscious:" With the above definitions for the action on U and $U^\dagger$ in the q and $\bar q$ representation, it automatically follows from representation theory that acting on U or $U^\dagger$ in an arbitrary representation produces analogous formulae with the generators appearing on the r.h.s. in that representation.

Where there are different U fields to distinguish, I will write $\nabla^q_{U_{\boldsymbol{x}}}$ instead of $\nabla^a_{\boldsymbol{x}}$ for definiteness.

These definitions and properties will become the core technical tool in the generalization of the BMS equation that is to follow below. For the time being, let me note that using both of these objects, $H_{\rm FP}$ is elegantly written as

$$H_{\rm FP} = -\frac{1}{2}\frac{\alpha_s}{\pi^2}\, \mathcal{K}_{\boldsymbol{xzy}} \left[i\nabla^a_x i\nabla^a_y + i\bar\nabla^a_x i\bar\nabla^a_y + \tilde U^{ab}_z (i\bar\nabla^a_x i\nabla^b_y + i\nabla^a_x i\bar\nabla^b_y)\right] \tag{26}$$

[2]Note that nevertheless, both correspond to what is called a left action of the group, as $f(e^{-i\omega}e^{-i\eta}, U) = f(e^{-i\omega}, f(e^{-i\eta}, U))$ in both cases.

[integration convention for x, z, y]. While the factorized form of Eq. (19) is most useful in a derivation of a Langevin description of the evolution that allows a numerical implementation, this second form is more economical in the derivation of evolution equations for given correlators that follow as a consequence from Eq. (18).

Among these, the equation for the two point operator $\hat{S}_{\boldsymbol{xy}}$ is what I will turn to next in order to make contact with the BK equation. Multiplying both sides of Eq. (18) with $\hat{S}_{\boldsymbol{xy}}$ and taking the average over U according to Eq. (17), one immediately arrives at

$$\begin{aligned}\partial_\tau \langle \hat{S}_{\boldsymbol{xy}} \rangle_\tau =& \frac{\alpha_s}{2\pi^2} \Big\langle \Big[\mathcal{K}^{(1)}_{\boldsymbol{uzv}} \Big(i\nabla^a_{\boldsymbol{u}} i\nabla^a_{\boldsymbol{v}} + i\tilde{\nabla}^a_{\boldsymbol{u}} i\tilde{\nabla}^a_{\boldsymbol{v}} \Big) + \mathcal{K}^{(2)}_{\boldsymbol{uzv}} \Big(\big[U_{\boldsymbol{z}}\big]^{ab} (i\tilde{\nabla}^a_{\boldsymbol{u}} i\nabla^b_{\boldsymbol{v}} + i\tilde{\nabla}^a_{\boldsymbol{v}} i\nabla^b_{\boldsymbol{u}}) \Big) \Big] \frac{\mathrm{tr}(U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}})}{N_c} \Big\rangle_\tau \\ =& \frac{\alpha_s}{2\pi^2} \Big\langle \Big[\big(2\mathcal{K}^{(1)}_{\boldsymbol{xzy}} - \mathcal{K}^{(1)}_{\boldsymbol{xzx}} - \mathcal{K}^{(1)}_{\boldsymbol{yzy}} \big) \Big(-2C_{\mathrm{f}} \frac{\mathrm{tr}(U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}})}{N_c} \Big) \\ & + \big(2\mathcal{K}^{(2)}_{\boldsymbol{xzy}} - \mathcal{K}^{(2)}_{\boldsymbol{xzx}} - \mathcal{K}^{(2)}_{\boldsymbol{yzy}} \big) 2 \big[U_{\boldsymbol{z}}\big]^{ab} \frac{\mathrm{tr}(t^a U_{\boldsymbol{x}} t^b U^\dagger_{\boldsymbol{y}})}{N_c} \Big] \Big\rangle_\tau \, . \end{aligned} \tag{27}$$

In these expressions I have taken pains to label the contributions from the terms with and without an additional factor of $\tilde{U}_{\boldsymbol{z}}$ by $^{(1)}$ and $^{(2)}$ for ease of reference. In the JIMWLK case there is no distinction between the kernels in the two cases and one finds that the linear combinations of JIMWLK kernels just assemble into the BK kernel:

$$\tilde{\mathcal{K}}_{\boldsymbol{xzy}} = \big(2\mathcal{K}^{(i)}_{\boldsymbol{xzy}} - \mathcal{K}^{(i)}_{\boldsymbol{xzx}} - \mathcal{K}^{(i)}_{\boldsymbol{yzy}} \big) \tag{28}$$

in both terms. To compare with the factorizing limit, one applies the Fierz identity

$$\big[U_{\boldsymbol{z}}\big]^{ab} \mathrm{tr}(t^a U_{\boldsymbol{x}} t^b U^\dagger_{\boldsymbol{y}}) = \frac{1}{2} \Big(\mathrm{tr}(U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{z}}) \mathrm{tr}(U_{\boldsymbol{z}} U^\dagger_{\boldsymbol{y}}) - \frac{1}{N_c} \mathrm{tr}(U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}) \Big) \tag{29}$$

to rewrite Eq. (27) as

$$\partial_\tau \langle \hat{S}_{\boldsymbol{xy}} \rangle_\tau = \frac{\alpha_s N_c}{2\pi^2} \int d^2z \; \tilde{\mathcal{K}}_{\boldsymbol{xzy}} \, \langle \hat{S}_{\boldsymbol{xz}} \hat{S}_{\boldsymbol{zy}} - \hat{S}_{\boldsymbol{xy}} \rangle_\tau \, . \tag{30}$$

This immediately reduces to Eq. (30) if one assumes factorization in the nonlinearity on the r.h.s. by replacing

$$\langle \hat{S}_{\boldsymbol{xz}} \hat{S}_{\boldsymbol{zy}} \rangle_\tau \to S_{\tau;\boldsymbol{xz}} \, S_{\tau;\boldsymbol{zy}} \, . \tag{31}$$

Without this assumption, the evolution of the two point function contains a three point function, which in turn will couple to yet higher orders via an infinite hierarchy fully contained in Eq. (18). The reason for this is the presence of $\tilde{U}_{\boldsymbol{z}}$ in Eq. (26). It is important to note that the factorization assumption (31) decouples and factorizes the whole hierarchy. Note that the real emission term originates solely from the contributions labelled (2) in Eq. (27) while the virtual corrections receive contributions from both.

Since it would appear that the factorization assumption is closely linked with the large N_c limit, it would appear that if an interpretation of $G_{ab}(E)$ in terms of averages over eikonal factors in analogy with $S_{\tau;\boldsymbol{xy}}$ is possible, these objects should satisfy an evolution equation that is in close analogy to the JIMWLK equation. This is the premise on which I base the conjecture for the finite N_c generalization of the BMS equation. To write it down I need

1. to use the correspondences of variables and kernels noted in the above. In this step it proves useful, if I slightly redefine the functional aspect of the invariant vector fields to produce δ functions adapted to the structure imposed by the solid angle measure. To this end I will write

$$i\nabla^a_p U_p = t^a U_p \bar{\delta}(p-q) \tag{32}$$

where $\bar{\delta}(p-q)$ contains Jacobian factors such that

$$\int \frac{d\Omega_k}{4\pi} \bar{\delta}(p-q) f(q) = f(p) \, . \tag{33}$$

This will simplify expressions considerably.

2. More importantly, I need to take into account the inside-outside distinction of the jet geometry by allowing the counterparts of $\mathcal{K}^{(i)}$ to be different for $i = 1, 2$. Matching can then be done on the BMS level by considering the evolution equation for the two point function suggested in (15). This is easily done and amounts to an algebraic exercise that fully determines both kernels from a rederivation of the BMS equation from its generalization via exactly the same steps displayed above for the JIMWLK/BK pair. Clearly both analogues of $\mathcal{K}^{(i)}$ will be proportional to $w_{pq}(k)$ with the proportionality factor $f^{(i)}(k)$ carrying the jet geometry via some $u(k)$ dependence.[3]

The result of this exercise is a Fokker-Planck Hamiltonian I will dub $H_{\rm ng}$ (for "non-global"). I will present it in two forms, to parallel the two versions of the BMS equation, (3) and (4). To this end I will introduce an additional function $\tilde{f}^{(2)}(k)$ that will serve as the coefficient of the generalization of the Sudakov radiator term. One has the following forms of the Hamiltonian:

$$H_{\rm ng} := -\frac{\alpha_s}{2\pi}\, w_{uv}(k)\Big[f^{(1)}(k)\big(i\nabla^a_{\boldsymbol{u}} i\nabla^a_{\boldsymbol{v}} + i\bar\nabla^a_{\boldsymbol{u}} i\bar\nabla^a_{\boldsymbol{v}}\big) + f^{(2)}(k)\big[U_{\boldsymbol{k}}\big]^{ab}\big(i\bar\nabla^a_{\boldsymbol{u}} i\nabla^b_{\boldsymbol{v}} + i\bar\nabla^a_{\boldsymbol{v}} i\nabla^b_{\boldsymbol{u}}\big)\Big] \tag{34a}$$

in analogy with (3), and

$$\begin{aligned} H_{\rm ng} := -\frac{\alpha_s}{2\pi}\, w_{uv}(k)\Big[&\tilde{f}^{(1)}(k)\Big(i\nabla^a_{\boldsymbol{u}} i\nabla^a_{\boldsymbol{v}} + i\bar\nabla^a_{\boldsymbol{u}} i\bar\nabla^a_{\boldsymbol{v}}\Big) \\ &+ f^{(2)}(k)\Big(i\nabla^a_{\boldsymbol{u}} i\nabla^a_{\boldsymbol{v}} + i\bar\nabla^a_{\boldsymbol{u}} i\bar\nabla^a_{\boldsymbol{v}} + \big[U_{\boldsymbol{k}}\big]^{ab}(i\bar\nabla^a_{\boldsymbol{u}} i\nabla^b_{\boldsymbol{v}} + i\bar\nabla^a_{\boldsymbol{v}} i\nabla^b_{\boldsymbol{u}})\Big)\Big] \end{aligned} \tag{34b}$$

to parallel (4). In both cases u, v and k are integrated over according to an "integration convention" that uses $\frac{d\Omega_p}{4\pi}$ as its measure for any of the momenta. Retracing the steps leading from the definition of the JIMWLK equation to (30) by eye should make it obvious that indeed, the separation of terms in Eq. (34b) is such that the first line generates the Sudakov radiator and the second the nonlinear evolution inside the jet cones.

A comparison of the result with BMS allows one to determine only the leading N_c part of the $f^{(i)}$. Already the $1/N_c$ corrections are not controlled by the matching. To this accuracy the f are N_c independent:

$$f^{(2)}(k) = u(k) \tag{35a}$$

$$f^{(1)}(k) = 1 \tag{35b}$$

$$\tilde{f}^{(1)}(k) = (1 - u(k))\ . \tag{35c}$$

Actually, it would be unnatural for the Fokker-Planck Hamiltonian to contain any explicit N_c dependence. If this were the case, it would require very special circumstances for it to incorporate *all* finite N_c corrections to BMS in a consistent manner. For this reason I take Eqns. (34), (35) together with a Fokker-Planck equation of the form

$$E\partial_E \hat{Z}_E[U] = -H_{\rm ng}\hat{Z}_E[U] \tag{36}$$

to define my conjecture for the finite N_c generalization of the BMS equation. This result will, of course, be derived independently below.

The most important feature of this equation is its infrared finiteness which directly relates to the real virtual cancellations being correctly encoded in Eq. (34b). Considering the Sudakov term in $H_{\rm ng}$ one finds that just as in the BMS limit, the phase space integration over k is restricted to the outside region where no hard particles are to be found. Therefore there is no divergence from the poles in $w_{uv}(k)$. The second term may be recast in analogy to Eq. (19) and the same finiteness argument used there clearly carries over to this situation. By looking at the BMS limit one sees that this same cancellation of terms corresponds to a real-virtual cancellation in the conventional sense. On the functional side, real and virtual contributions arise from different terms. Contributions with an additional factor $\tilde{U}_k$ correspond to real emission, the rest to virtual corrections.

The idea to interpret jet transition probabilities as correlators of path-ordered exponentials in singlet projections properly takes into account gauge invariance, just the same way as the formulation of the JIMWLK

[3]Note that the calculations are somewhat simplified by the fact that $w_{pp}(k) = 0$. The relation between kernels on the functional side to those on the BMS side as given by Eq. (28) for JIMWLK/BK pair, becomes a matter of simple proportionality.

equation does. The equation suggested therefore would appear to consistently encode the physics it is intended to cover with a set of very strong constraints. From this perspective it would be rather surprising would it not be possible to arrive at this result from first principles.

4 An equivalent Langevin description

Translation of a Fokker-Planck equation to an equivalent Langevin description is a textbook topic, that for the present case is only complicated by the functional nature of the equation and the group valued nature of variables involved. That such a step is possible in the JIMWLK case has been first suggested in [16] and worked out in detail in [22].

Analytical results from the Fokker-Planck formulation are generically hard to come by, although some information on the fixed point structure has been obtained in the case of the JIMWLK equation. This makes the alternative formulation even more useful, as it paves the way for a numerical implementation of the evolution equation. This has already been successfully carried out in the JIMWLK case in [19].

The Langevin formulation abandons the description in terms of weight functionals $\hat{Z}_\tau$ in favor of one in terms of ensembles of fields which are governed by the corresponding Langevin equations. Explicitly, to calculate any observable $O[U]$ of the fields U one writes

$$\langle O[U]\rangle_\tau = \int \hat{D}[U] O[U] \hat{Z}_\tau[U] \approx \frac{1}{N} \sum_{U\in \mathsf{E}[\hat{Z}_\tau]} O[U] \tag{37}$$

where, separately at each τ, the sum is over an ensemble $\mathsf{E}[\hat{Z}_\tau]$ of N configurations U. Its members can be thought of as created randomly according to the distribution $\hat{Z}_\tau$. Clearly, for $N\to\infty$, the ensemble and $\hat{Z}_\tau$ contain the same information.

The dynamics are then encoded in a Langevin equation, a finite difference equation in evolution "time" τ that contains a random force driven by a noise Ξ.

Prerequisites are (termwise) positive definiteness of the Fokker-Planck Hamiltonian in question. Only then can one introduce bounded noise integral – if needed separately for each positive term. If the Hamiltonian can be (termwise) separated into two conjugate factors in analogy to the structure of Eq. (19), any correlation effects can be included in the Langevin equation. The noise can then be taken to be (termwise) uncorrelated. To establish the structure one may expect for the result, let me give the corresponding expressions for the JIMWLK equation. There, the Langevin equation schematically[4] reads

$$\partial_\tau \, [U_{\tau;\boldsymbol{x}}]_{ij} = [U_{\tau;\boldsymbol{x}} i t^a]_{ij} \Big[\int d^2y \, \mathcal{E}^{ab;k}_{\boldsymbol{xy}}[U_\tau] \xi^{b,k}_{\tau;\boldsymbol{y}} + \hat{\sigma}^a_{\boldsymbol{x}}[U_\tau] \Big] \tag{38}$$

where

$$\mathcal{E}^{ab;k}_{\boldsymbol{xy}}[U_\tau] = \Big(\frac{\alpha_s}{\pi^2}\Big)^{1/2} \frac{(\boldsymbol{x}-\boldsymbol{y})_k}{(\boldsymbol{x}-\boldsymbol{y})^2} [1 - \tilde{U}^\dagger_{\tau;\boldsymbol{x}} \tilde{U}_{\tau;\boldsymbol{y}}]^{ab} \tag{39}$$

is the "square root" of χ, $\chi^{ab}_{\boldsymbol{xy}} = \mathcal{E}^{ac}_{\boldsymbol{xz}} \mathcal{E}^{cb}_{\boldsymbol{zy}}$. Factorization is complete and no termwise split is necessary (or even possible without violating positivity). σ is given by

$$\hat{\sigma}^a_{\boldsymbol{x}}[U_\tau] := \frac{1}{2} \nabla^b_{\boldsymbol{y}} \hat{\chi}^{ab}_{\boldsymbol{xy}} = i\big(\frac{1}{2}\frac{\alpha_s}{\pi^2} \int d^2z \mathcal{K}_{\boldsymbol{xzx}} \tilde{\mathrm{tr}}(\tilde{t}^a \tilde{U}^\dagger_{\tau;\boldsymbol{x}} \tilde{U}_{\tau;\boldsymbol{z}})\big) \, . \tag{40}$$

The ξ are independent Gaussian random variables with correlators determined according to

$$\langle \ldots \rangle_\xi = \int D[\xi] \, (\ldots) \, e^{-\frac{1}{2}\xi\xi} \, . \tag{41}$$

[4]This is a continuous time version of the equation that strictly speaking is not unique. The path-integral derivation shown in [22] and maybe more transparently in the appendix of [19] makes it clear that we are to take a "retarded" prescription here in which the derivative on the l.h.s is taken as a finite difference and the fields on the r.h.s. are determined at the previous time step.

Fortunately all of the prerequisites listed above are also met for $H_{\rm ng}$. In order to achieve termwise factorization I first separate off the $f^{(2)}$ terms and further treat the two contributions proportional to $\tilde{f}^{(1)}$ separately. All of these are positive due to the definition of $u(k)$:

$$u(k) = \Theta_{\rm in}(k) + e^{-\nu\omega}\Theta_{\rm out}(k) \tag{42}$$

ensures $0 < u(k) < 1$ and the same then for the coefficient functions in Eq. (34b).

Correspondingly one would write the Hamiltonian on the JIMWLK level as

$$H_{\rm ng} = \frac{1}{2} i\nabla_u^a \chi_{uv}^{ab} i\nabla_v^b \tag{43}$$

where u, v are integrated over with $\frac{d\Omega_u}{4\pi}\frac{d\Omega_v}{4\pi}$ and

$$\chi_{uv}^{ab} = -\int \frac{d\Omega_k}{4\pi}\frac{\alpha_s}{\pi} w_{uv}(k)\Big[\tilde{f}^{(1)}(k)(1 + U_u^\dagger U_v) + f^{(2)}(k)(1 - U_u^\dagger U_k)(1 - U_k^\dagger U_v)\Big]^{ab} . \tag{44}$$

Factorization of χ according to $\chi = \mathcal{E}\mathcal{E}^\dagger$ is achieved by defining a three component structure representing the three separately positive terms alluded to before:

$$\mathcal{E}_{pk}^{ab;\mu} = \sqrt{\frac{\alpha_s}{\pi}}\frac{p^\mu}{p.k}\{\sqrt{\tilde{f}^{(1)}(k)}\delta^{ab}, \sqrt{\tilde{f}^{(1)}(k)}[U_p^\dagger]^{ab}, \sqrt{f^{(1)}(k)}(1 - U_p^\dagger U_k)^{ab}\} . \tag{45}$$

Correspondingly one has to introduce independent white noise for all of the components,

$$\Xi_k^{b;\mu} = \{(\xi^{(1)})_k^{b;\mu}, (\xi^{(1')})_k^{b;\mu}, (\xi^{(2)})_k^{b;\mu}\} . \tag{46}$$

The only thing left to cope with is the measure and the δ-functions in the noise correlator, which need to be such that

$$\big\langle \big(\int \frac{\Omega_k}{4\pi}\mathcal{E}_{pk}^{ac;\mu}\Xi_k^{c;\mu}\big)\big(\int \frac{\Omega_l}{4\pi}\mathcal{E}_{ql}^{ad;\nu}\Xi_l^{d;\nu}\big)\big\rangle = \chi_{pq}^{ab} . \tag{47}$$

Now the measure is $\frac{d\cos\theta d\phi}{4\pi}$ and one needs the correlators to read

$$\langle (\xi^i)_p^{a;\mu}(\xi^j)_q^{b;\nu}\rangle = 4\pi\delta(\cos\theta_p - \cos\theta_q)\delta(\phi_p - \phi_q)\delta^{ij}g^{\mu\nu} . \tag{48}$$

With these preparations and the identification $\tau \leftrightarrow \ln E$ one has a Langevin equation that reads

$$\partial_\tau\,[U_{\tau;p}]_{ij} = [U_{\tau;p} it^a]_{ij}\Big[\int \frac{d\Omega_k}{4\pi}\,\mathcal{E}_{pk}^{ab;\mu}[U_{\tau;p}]\;\Xi_{\tau;k}^{b;\mu}\Big] . \tag{49}$$

In comparison with the JIMWLK case, any sigma-terms vanish because $w_{pq}(k)$ satisfies $w_{pp}(k) = 0$.

Note again that any continuum notation is deceptive. The equation to consider is really a finite difference equation and upon iteration will allow for an interpretation of subsequent, ordered soft gluon emission. To this end it is instructive to rewrite the Langevin equation in terms of functional derivatives:

$$\partial_\tau\,[U_{\tau;p}]_{ij} = [U_{\tau;p} it^a]_{ij}\Big[\int \frac{d\Omega_k}{4\pi}\,\mathcal{E}_{pk}^{ab;\mu}[U_{\tau;p}]\;\Xi_{\tau;k}^{b;\mu}\Big] = \Big\{\int \frac{d\Omega_q}{4\pi}\frac{d\Omega_k}{4\pi}\; i\mathcal{E}_{pq}^{ab;\mu}[U_{\tau;p}]\;\Xi_{\tau;k}^{b;\mu} i\nabla^a_{U_{\tau;q}}\Big\}[U_{\tau;p}]_{ij} . \tag{50}$$

The operator in the last version contains two terms in its third component, the component responsible for the in-region. Using a somewhat more compact notation for the τ dependence they read:

$$\int \frac{d\Omega_q}{4\pi}\int \frac{d\Omega_k}{4\pi}\,\Big[i(\mathcal{E}^3)_{pq}^{ab;\mu}\;\Xi_k^{b;\mu} i\nabla_q^a\Big]_{\tau'} = \int \frac{d\Omega_q}{4\pi}\frac{d\Omega_k}{4\pi}\, i\sqrt{\frac{\alpha_s}{\pi}}\frac{p^\mu}{p.q}\Big[(i\nabla_p^a + \tilde{U}_q^{ba} i\bar{\nabla}_p^b)\;\Xi_k^{a;\mu}\Big]_{\tau'} . \tag{51}$$

Taking into account the fact that the Langevin description is defined only in a τ-discretized sense, one might be tempted to conclude that the terms containing factors $\tilde{U}_q$ correspond to real gluon emission, while the others would generate virtual corrections, not only in this, but also the other components.

A formal solution to (49) is

$$[U_{\tau,p}]_{ij} = P_\tau \exp\Big\{ \int_{\tau_0}^{\tau} d\tau' \int \frac{d\Omega_q}{4\pi}\frac{d\Omega_k}{4\pi}\, i\Big[\mathcal{E}^{ab;\mu}_{pq}\ \Xi^{b;\mu}_k i\nabla^a_q\Big]_{\tau'}\Big\}[U_{\tau_0,p}]_{ij}\ , \tag{52}$$

which again is to be interpreted in a discrete sense. To write this expression I have again adapted the definition of the functional derivatives to include a δ-function in τ: $i\nabla^a_{\tau,p}U_{\tau' q} = -U_{\tau,p}t^a\tilde{\delta}(p-q)\delta(\tau-\tau')$.

This would have the interpretation of subsequent real emission with virtual corrections correctly taken into account.

Instead of jumping to conclusions at this point, I will substantiate these ideas with a derivation from first principles that follows the strategy of BMS. For this one needs a formulation in terms of functionals capable of handling finite N_c corrections. This I will turn to next. While doing so, a number of interesting structures, like a closed, finite N_c version of Eq. (8) will emerge alongside similar expressions for the underlying amplitudes.

5 Amplitudes in the strongly ordered domain

In order to find a suitable starting point to go beyond the large N_c limit one has to go rather far back and start with a general discussion of soft gluon amplitudes along the lines given already in [20, 21].

To help organize the argument, I will start with the definition of a generating functional for the (tree level) amplitudes of (real) soft gluon emission from a $q\bar{q}$ pair in the strongly ordered region

$$\omega_{k_n} \ll \omega_{k_{n-1}} \ll \dots \omega_{k_1} \ll \omega_p < \omega_q \tag{53}$$

written as

$$\mathsf{A}^{ij}_{p\,q}[\Xi] := \sum_{n=0}^{\infty}\int \frac{d\Omega_{k_1}}{4\pi}\dots\frac{d\Omega_{k_n}}{4\pi} A(^{ija_1\dots a_n}_{qpk_1\dots k_n})\Xi^{a_1}_{k_1}\dots\Xi^{a_n}_{k_n}\ . \tag{54}$$

The amplitudes for n-gluon emission from a initial $q\bar{q}$ pair are denoted by $A(^{ija_1\dots a_n}_{qpk_1\dots k_n})$, where the momenta $q, p, k_1 \dots k_n$ and color indices $i, j, a_1 \dots a_n$ are explicitly listed, a corresponding set of Lorentz-indices $\mu_1 \dots \mu_n$ is suppressed. They are isolated by n-fold (functional) differentiation w.r.t. Ξ at $\Xi = 0$. At this stage, Ξ is just an external source, the relationship to the noise of the Langevin description will become clear later.

The $A(^{ija_1\dots a_n}_{qpk_1\dots k_n})$ are known to satisfy an iterative structure in which the color singlet $\to q\bar{q}g^{n+1}_{\rm soft}$-amplitude follows by induction from the corresponding amplitude with n soft gluons. As a result, the 0-gluon term in Eq. (54) fully determines the whole functional, although the explicit construction of the n-th order expression with the tools available to date becomes more and more cumbersome with growing n. The rules of the game have been explicitly demonstrated in [21] and require no recourse to the $1/N_c$ limit.

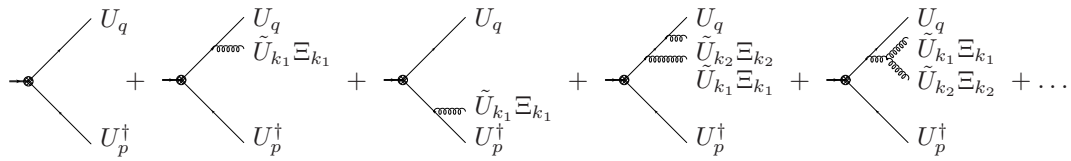
I will now try to establish a method that will allow me to give a closed operator form of the full tower of amplitudes contained in Eq. (54). While explicit generation of a term with arbitrary n will still be cumbersome, the fact that I have a closed mathematical expression instead of a prescription for an iterative procedure will allow me to use it in calculations and eventually in the derivation of the evolution equation.

For this purpose, a slight generalization of the definition (54) is operationally somewhat more useful in that it allows me to specify an explicit functional form that casts the iteration step of [21] as a simple functional operation. This operation will involve nothing more complicated than the functional differentiation rules given in Eq. (24) and a suitably defined $n = 0$ term to start the process of generating the amplitudes.

The starting point is the definition

$$\mathsf{A}^{ij}_{p\,q}[U,\Xi] := \sum_{n=0}^{\infty}\int \frac{d\Omega_{k_1}}{4\pi}\dots\frac{d\Omega_{k_n}}{4\pi} A(^{\tilde{i}\tilde{j}a_1\dots a_n}_{qpk_1\dots k_n})\tilde{U}^{a_1b_1}_{k_1}\Xi^{b_1}_{k_1}\dots\tilde{U}^{a_nb_n}_{k_n}\Xi^{b_n}_{k_n}\ [U^\dagger_p]_{i\tilde{i}}[U_q]_{\tilde{j}j}\ . \tag{55}$$

Here I have simply added a factor of U in the appropriate representation to each leg of the diagrams contained in the definition of the ordered amplitudes. Diagrammatically the first few contributions read:

$\mathsf{A}^{ij}_{p\,q}[U,\Xi] =$ 

(56)

Obviously Eq. (54) is just the special case at $U = 1$: $\mathsf{A}^{ij}_{p\,q}[\Xi] = \mathsf{A}^{ij}_{p\,q}[1,\Xi]$. The rationale behind this definition lies in the very nature of soft gluon emission underlying the construction of the amplitudes. Per definition, soft gluon emission does not change the direction of the parent (the emitter), in that role as a parent the entity (be it a quark or a gluon) is therefore written as a path ordered exponential along the direction of some momentum. Emission of a softer gluon, that in later steps will also serve as a parent will then necessarily add an adjoint factor $\tilde{U}_k$ and an eikonal emission vertex $J^\mu_{lk}t^a$ (with $J^\mu_{pk} := \frac{p^\mu}{p.k}$), where the generator is to be taken in the representation of the emitting object. As all these objects are encoded as eikonal lines U, this will automatically occur if one writes it via invariant vector fields $i\bar{\nabla}^a_l$ just as in the rewrite of the Langevin equation at the end of Sec. 4. In functional form, one ends up with the single emission operator

$$\int \frac{d\Omega_l}{4\pi}\frac{d\Omega_k}{4\pi}\, g\, J^\mu_{lk}\tilde{U}^{ab}_k\ \Xi^{b;\mu}_k i\bar{\nabla}^a_l\ . \tag{57}$$

Note how closely this resembles the real emission part of the Langevin equation (49). Eq. (55) should then be completely determined by the $n = 0$ term in the sum which describes $q\bar{q}$-pair without additional soft gluons: higher orders should just follow from repeated application of (57). This needs now to be checked against what is known about the soft gluon amplitudes.

First one needs to give an initial condition for this iteration, the bare $q\bar{q}$ term. This reads

$[\mathsf{A}^{(0)}]^{ij}_{p\,q}[U,\Xi] =$ 

$:= M_2(q,p)[U^\dagger_p U_q]_{ij}\ ,$ (58)

and corresponds to a zero order amplitude of the form $A(^{ij}_{qp}) = M_2(q,p)\delta^{ij}$[5].

The above prescription then claims that the $q\bar{q}g_{\rm soft}$ term is given by

$$[\mathsf{A}^{(1)}]^{ij}_{p\,q}[U,\Xi] = \Big\{\int \frac{d\Omega_l}{4\pi}\frac{d\Omega_k}{4\pi}\ g\, J^\mu_{lk}\tilde{U}^{ab}_k\ \Xi^{b;\mu}_k i\bar{\nabla}^a_l\Big\}[\mathsf{A}^{(0)}]^{ij}_{p\,q}[U,\Xi]\ . \tag{59}$$

To demonstrate once how to use the machinery, I will go through the steps explicitly. First insert $[\mathsf{A}^{(0)}]^{ij}_{p\,q}[U,\Xi]$ from (58). This yields

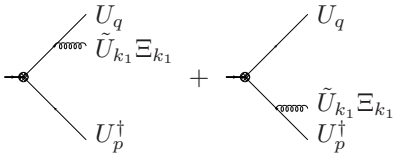
$$[\mathsf{A}^{(1)}]^{ij}_{p\,q}[U,\Xi] = \Big\{\int \frac{d\Omega_l}{4\pi}\frac{d\Omega_k}{4\pi}\ g\, J^\mu_{lk}\tilde{U}^{ab}_k\ \Xi^{b;\mu}_k i\bar{\nabla}^a_l\Big\}M_2(q,p)[U^\dagger_p U_q]_{ij}\ . \tag{60}$$

All that is left to do is to use the differentiation rules (24) (with the adaptation to phase space (65) taken into account) and differentiate the two factors $U^\dagger_p$ and U_q. The variation eliminates the l integral and one is

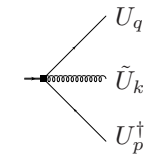
[5] For M_n I have adopted the notation of [20, 21]

left with

$$[\mathsf{A}^{(1)}]^{ij}_{p\,q}[U,\Xi] =$$



$$= \int \frac{d\Omega_k}{4\pi} M_2(p,q) A^\mu_{pq}(k) [U_p^\dagger t^a U_q]_{ij} U_k^{ab} \Xi_k^{b;\mu} =$$



(61)

where

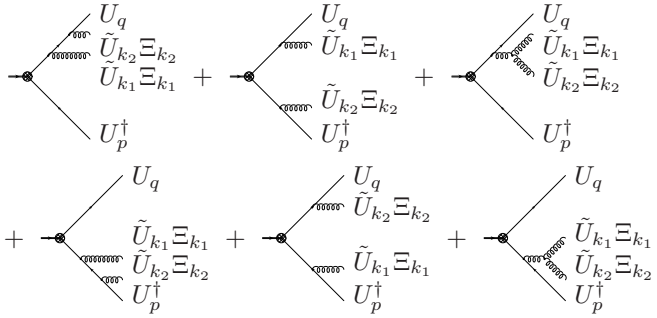
$$A^\mu_{pq}(k) = J^\mu_{pk} - J^\mu_{qk} \ . \tag{62}$$

Now one isolates the amplitude via a variation in Ξ and reads off

$$A^{ija}_{pqk}) = M_2(p,q) A^\mu_{pq}(k) [t^a]_{ij} =: M_3(p,k,q)[t^a]_{ij} \tag{63}$$

with $M_3(q,k,p)$ again as in the notation of [20, 21]. Eq. (63) in fact is identical to Eq. (40) of [21] and an additional gluon emission, i.e. iteration with Eq. (57), leads to their Eq. (42) for the amplitude with 2 soft gluons:

$$[\mathsf{A}^{(2)}]^{ij}_{p\,q}[U,\Xi] =$$



(64)

(The color algebra for the explicit comparison is given in App. A, case $n = 1$.) The pattern simply follows the rules of functional differentiation and a comparison of analytic expressions for the amplitudes as given by the iteration rules of [21] was performed explicitly using symbolic algebra tools up to $n = 6$, where the task starts to become time consuming. To compare beyond finite orders, a few additional tools are required.

In the above I have implemented the ordering "by hand," simply by requesting the hierarchy (53) for the ω_{k_i}. To write the all orders expression advertised above, I need one more ingredient: the addition of an energy integral to the emission operator (57) and an appropriate extension of the definition of $i\bar{\nabla}^a_p$ to $i\bar{\nabla}^a_{\omega_p,p}$ defined such that

$$i\bar{\nabla}^a_{\omega_p,p} U_q = t^a U_p \tilde{\delta}(p-q) \tag{65}$$

where $\tilde{\delta}(p-q)$ is adapted to the measure used in the momentum and energy integrations. The natural choice for the latter is to replace the solid angle integrations in (57) by full fledged phase space integrations:

$$\frac{d\Omega_k}{4\pi} \to d\Phi_k := \frac{d^4k}{(2\pi)^4}\delta(k^2)\theta(\omega_k) = \frac{\omega_k d\omega_k}{(2\pi)^2}\frac{d\Omega_k}{4\pi}\theta(\omega_k) \ . \tag{66}$$

$\tilde{\delta}(p-q)$ is then defined as the appropriate δ function on the forward light cone, i.e. with Jacobian factors such that

$$\int d\phi_k \tilde{\delta}(p-k) f(k) = f(p) \ . \tag{67}$$

(Note again the similarity of the roles played by τ-ordering in the Langevin description and energy ordering in this case.) With these definitions it is straightforward to verify that a replacement of the solid angle

integrations in the above by phase space integrations leads to no change in the previous results, if one implements energy ordering for subsequent definitions with ordering θ-functions, just as one would for the contour parameter in case of path ordered exponential. To leading logarithmic accuracy this will not change the result for the evolution equation.[6]

By now it should appear to be the natural choice if I write for the generating functional (again suppressing the ω-label for compactness)

$$\mathsf{A}^{ij}_{p\,q}[U,\Xi] = P_{\omega_k} \exp\Big\{ \int d\Phi_l d\Phi_k \; g\, J^{\mu}_{lk} \tilde{U}^{ab}_{k} \; \Xi^{b;\mu}_{k} i\bar{\nabla}^{a}_{l} \Big\} [\mathsf{A}^{(0)}]^{ij}_{p\,q}[U,\Xi] \tag{68}$$

where P_{ω_k} implements the strong ordering of Eq. (53). I should actually use a notation that displays the dependence on the functional form supplied in the zero emission part and for instance write $\mathsf{A}^{ij}_{p\,q}[U,\Xi,M_2(p,q)]$.

In analogy with path ordered exponentials, this implies that the exponential series is to be interpreted as

$$\begin{aligned} &P_{\omega_k} \exp\Big\{ \int d\Phi_k d\Phi_l \; g\, J^{\mu}_{lk} \tilde{U}^{ab}_{k} \; \Xi^{b;\mu}_{k} i\bar{\nabla}^{a}_{l} \Big\} \\ &:= \sum_{n=0}^{\infty} P_{\omega_k} \Big[\Big\{ \int d\Phi_{k_n} d\Phi_{l_n} \; g\, J^{\mu_n}_{l_n k_n} \tilde{U}^{a_n b_n}_{k_n} \; \Xi^{b_n;\mu_n}_{k_n} i\bar{\nabla}^{a_n}_{l_n} \Big\} \ldots \Big\{ \int d\Phi_{k_1} d\Phi_{l_1} \; g\, J^{\mu_1}_{l_1 k_1} \tilde{U}^{a_1 b_1}_{k_1} \; \Xi^{b_1;\mu_1}_{k_1} i\bar{\nabla}^{a_1}_{l_1} \Big\} \Big] \end{aligned} \tag{69}$$

where the $1/n!$ are absent due to the explicit ordering of the subsequent soft emissions. To allow for more compact formulae below, I will introduce the notation

$$\mathsf{U}[\Xi,U] := P_{\omega_k} \exp\Big\{ \int d\Phi_k d\Phi_l \; g\, J^{\mu}_{lk} \tilde{U}^{ab}_{k} \; \Xi^{b;\mu}_{k} i\bar{\nabla}^{a}_{l} \Big\} \tag{70}$$

for this functional differential operator. The explicit mention of the U-field in the notation will be needed later when different types of eikonal fields will appear. $\mathsf{U}[\Xi,U]$ will turn out to have a meaning independent of the bare ($n=0$) term it is used to act on. It will be called the (real) shower operator as it will show up wherever one is forming that ordered soft gluon cloud around a hard seed.

With these tools a direct comparison with the iterative scheme of [21] at all orders becomes possible. There it was shown (c.f. Eq. (45) of [21]) that the n-th order amplitude is of the form

$$A^{(ija_1\ldots a_n}_{qpk_1\ldots k_n)} = \sum_{\Pi_{n+2}(l)} M_n(q,k_{l_1},\ldots,k_{l_n},p)[t^{l_1}\ldots\ldots t^{l_n}]_{ij} \tag{71}$$

where $\Pi_{n+2}(l)$ denotes the set of permutations of the $\{p,l_1,\ldots,l_n,q\}$. Moreover, the amplitudes satisfy the recursion relation

$$A^{(ija_1\ldots a_n a_{n+1}}_{qpk_1\ldots k_n k_{n+1})} = \sum_{j=1}^{n} \sum_{\Pi_{n+2}(l)} M_n(q,k_{l_1},\ldots,k_{l_n},p) A_{l_j l_{j+1}}(k_{n+1})[t^{l_1}\ldots t^{l_j} t^{n+1} t^{l_{j+1}}\ldots t^{l_n}]_{ij} \tag{72}$$

(c.f. Eq. (48) of [21]). This determines all orders, once the first term is specified.

Since agreement has already been established for the first few terms ($n=0,1,2$ were compared explicitly, terms including $n=6$ have been checked), all that is left to show, is that the functional prescription given above leads to the recursion relation (72). This is readily achieved: Assuming the n-th order term, $[\mathsf{A}^{(n)}]^{ij}_{pq}[U,\Xi]$ to have the form (71), one simply uses a single soft gluon emission operator to find $[\mathsf{A}^{(n+1)}]^{ij}_{pq}[U,\Xi]$:

$$\begin{aligned} [\mathsf{A}^{(n+1)}]^{ij}_{p\,q}[U,\Xi] =& \Big\{ \int d\Phi_{k_{n+1}} d\Phi_{l_{n+1}} \; g\, J^{\mu_{n+1}}_{l_{n+1}k_{n+1}} \tilde{U}^{a_{n+1}b_{n+1}}_{k_{n+1}} \; \Xi^{b_{n+1};\mu_{n+1}}_{k_{n+1}} i\bar{\nabla}^{a_{n+1}}_{l_{n+1}} \Big\} [\mathsf{A}^{(n)}]^{ij}_{p\,q}[U,\Xi] \\ =& \Big\{ \int d\Phi_{k_{n+1}} d\Phi_{l_{n+1}} \; g\, J^{\mu_{n+1}}_{l_{n+1}k_{n+1}} \tilde{U}^{a_{n+1}b_{n+1}}_{k_{n+1}} \; \Xi^{b_{n+1};\mu_{n+1}}_{k_{n+1}} i\bar{\nabla}^{a_{n+1}}_{l_{n+1}} \Big\} \\ & \int d\Phi_{k_1}\ldots d\Phi_{k_n} A^{(\tilde{i}\tilde{j}a_1\ldots a_n}_{qpk_1\ldots k_n)} \tilde{U}^{a_1 b_1}_{k_1} \Xi^{b_1}_{k_1} \ldots \tilde{U}^{a_n b_n}_{k_n} \Xi^{b_n}_{k_n} \; [U_p^{\dagger}]_{i\tilde{i}} [U_q]_{\tilde{j}j} \; . \end{aligned} \tag{73}$$

[6] Also the results below for the structure of the multiple ordered soft emission can be obtained both ways – to have a formulation with all ingredients (including the ordering) written explicitly is merely a convenience from this perspective. What the formalism does, is to allow to give a closed expression for the generating functional, Eq. (68).

It is straightforward to carry out the differentiation: The product rule leads to a sum over j similar to that in Eq. (72). Then one uses $[\tilde{t}^{a_{n+1}}]^{b_{n+1}a_n}[t^{a_n}]_{kl} = [t^{n+1}, t^{a_n}]_{ij}$ to combine the J factors into the $A_{l_j l_{j+1}}(k_{n+1})$. After this rearrangement the sum only runs over $j = 1, \ldots, n$ instead of $p, 1, \ldots, n, q$ and matches (72). The algebraic details of this rearrangement are given in App. A. This shows that one indeed creates the same amplitudes with the *full* color structure, i.e. without any recourse to the $1/N_c$ approximation.

While in the above, I have done nothing more but to provide a functional form of a (real) ordered soft emission amplitude, I will now show that there is structure contained in the above that is very similar to the information contained in the Langevin equation and will eventually lead to the RG equation stated earlier. The main observation in this regard is that, as a consequence of the strong ordering, any $\frac{\delta}{\delta\Xi_k^{a;\mu}}$ in which k is at (or near) the phase space boundary, there will be only a contribution from the first emitted gluon ($n = 1$). As a consequence, for such variations one gets

$$\frac{\delta}{\delta\Xi_k^{b;\mu}}\mathsf{U}[\Xi, U] = \mathsf{U}[\Xi, U]\Big\{ \int d\Phi_l \; J^\mu_{lk}\tilde{U}^{ab}_k i\bar{\nabla}^a_l \Big\} \; . \tag{74}$$

Note that

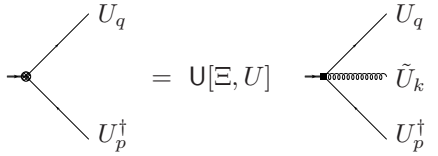
$$\mathsf{A}^{ijb}_{p\,q\,k}[U, \Xi, M_3(p, k, q)] := \mathsf{U}[\Xi, U] \; \tilde{U}^{ab}_k [U^\dagger_p t^a U_q]_{ij} M_3(p, k, q) \tag{75}$$

is the $q\bar{q}g$ counterpart to Eq. (68). In short, Eq. (74) states, that under variation at the phase space boundary, one relates the generating functional for a given tower of amplitudes to that of a tower with an additional hard gluon. For the above example it relates the amplitudes with a hard $q\bar{q}$ to those with hard $q\bar{q}g$ content:

$$\frac{\delta}{\delta\Xi_k^{b;\mu}}\mathsf{A}^{ij}_{p\,q}[U, \Xi, M_2(p, q)] = \mathsf{A}^{ijb}_{p\,q\,k}[U, \Xi, A^\mu_{p\,q}(k) M_2(p, q)] \; , \tag{76}$$

or diagrammatically

$$\frac{\delta}{\delta\Xi_k^{b;\mu}}\,\mathsf{U}[\Xi, U] \quad \begin{array}{l} \\ \\ \\ \end{array} \quad = \quad \mathsf{U}[\Xi, U] \quad \begin{array}{l} \\ \\ \\ \end{array} \quad . \tag{77}$$



These two items, the ordered nature leading to (74) under "hard" variations, and the fact that in this situation the number of hard legs of the soft emission amplitudes is increased by one as exemplified in (77), will be the core observations behind the evolution equation I am aiming at.

6 Transition probabilities from Ξ averages

As a first step to understand how to translate the above into expressions for transition probabilities, let me take the situation of a (real) $q\bar{q}g^n_{\text{soft}}$ soft emission amplitude in which one obtains the standard result

$$\begin{aligned} A(^{ija_1\ldots a_n}_{qpk_1\ldots k_n})[A(^{ija_1\ldots a_n}_{qpk_1\ldots k_n})]^\dagger = \sum_{\Pi_{n+2}(l)} \sum_{\Pi_{n+2}(l')} & M_n(q, q_{l_1}, \ldots, q_{l_n}, p) M_n(q, q_{l_1}, \ldots, q_{l_n}, p)^* \\ & \times \mathrm{tr}(t^{a_{l_1}} \ldots\ldots t^{a_{l_n}} t^{a_{l'_n}} \ldots\ldots t^{a_{l'_1}}) \; , \end{aligned} \tag{78}$$

in which I have labelled the gluon momenta in the final state by $q_1, \ldots, q_n$. The same expression is obtained, if one considers the Ξ averaged product of functionals

$$\langle [\mathsf{A}^{(n)}]^{ij}_{p\,q}[U, \Xi][\mathsf{A}^{(n)}]^{ij}_{p\,q}[U, \Xi]^\dagger \rangle_{n,\Xi} \tag{79}$$

if the average is defined with a Gaussian distribution for Ξ in which only strongly ordered modes $q_1, \ldots, q_n$ have support. I write

$$\langle \ldots \rangle_{n\Xi} := \det(M_n)^{\frac{1}{2}} \int D[\Xi] \ldots e^{-\frac{1}{2}\int d\phi_p d\phi_q \Xi^t_p M^{-1}_{n,\,pq} \Xi_q} \tag{80}$$

where $d\phi_k$ denotes the phase space integral for momentum k and I have suppressed discrete indices. M defines the Ξ correlator, which in the fully exclusive case considered above would read

$$M^{a,\mu\ b,\nu}_{n,\,pq} = \langle \Xi^{a,\mu}_p \Xi^{b,\nu}_q \rangle_{n,\Xi} = \delta^{ab} g^{\mu\nu} \tilde{\delta}(p-q) \sum_{i=1}^{n} \tilde{\delta}(q-q_i) \tag{81}$$

In this case, because of the strong ordering of momenta in the amplitudes, the Ξ integral will strictly *pair* off the one available momentum k_i in the amplitudes falling into the same range as q_i.

As such, this is only true if I have ordering in the emission vertices, either done by hand in the 2-d version, or via P_{ω_k} in the 2+1-d formulation. Only then, unwanted cross terms are excluded. The U factors just cancel trivially.

To make contact with the non-global observables of the BMS setting, one needs to depart from the fixed n situation and replace the $\sum_{i=1}^{n} \tilde{\delta}(q-q_i)$, which selects a given set of final state momenta in the case where the number of final state gluons is fixed, by a cutoff criterion that restricts real emission by geometry and energy. By writing

$$M^{a,\mu\ b,\nu}_{pq} = \langle \Xi^{a,\mu}_p \Xi^{b,\nu}_q \rangle_{\Xi} = \delta^{ab} g^{\mu\nu} \tilde{\delta}(p-q)\ \theta(E-\omega_q) u(q) \tag{82}$$

one can indeed go beyond a final state with a fixed number of gluons and consider

$$\langle \mathsf{A}^{ij}_{p\,q}[U,\Xi] \mathsf{A}^{ij}_{p\,q}[U,\Xi]^\dagger \rangle_{\Xi}\ , \tag{83}$$

which will provide the finite N_c-generalization of $G^{(\mathrm{real})}_{ab}$ of BMS.

In fact any degree of inclusiveness may be imposed by modifying the restrictions on the allowed modes appearing on the r.h.s. of Eq. (82).

At this point a few comments about the meaning of Ξ as it appears in the above are in order. Viewed from a diagrammatic point of view, the Ξ stand for nothing else but the final state soft gluons. This explains why there is an additional factor of i together with the use of the free gluon phase space measure in the averaging, compared to structures encountered in Sec. 4. There, Ξ should be viewed simply as a perturbative gluon and the same ingredients as those encountered here would show up in the formulation of S-matrix elements via the amputation of external legs.

This realization should make it clear how to include virtual corrections right from the outset, by a fairly simple modification of the above. I will refrain from doing so here and instead keep a closer parallel with the strategy employed by [1, 20, 21] and use the expressions for real emission to deduce the structure of the evolution equation.

7 Evolution equations for soft semi-inclusive quantities

For semi-inclusive quantities like the non-global observables of [1], in which one sums over a given, limited phase space volume of soft gluons, it is natural to ask for the dependence on that phase space boundary. Unlike a direct calculation of the average in Eq. (83) this should lead to a tractable result that exhibits new structure. In fact, I shall demonstrate that one recovers the generalization to the BMS equation suggested above.

To arrive at an evolution equation, I will simply take a derivative w.r.t. to the phase space boundary of the emitted gluons in the semi-inclusive probability (83). This is in direct correspondence with the procedure used in BMS. Here it is essential that one includes the phase space boundary into the definition of M. Then the result is most transparently displayed using the following simple relationship, based on functional differentiation and Legendre transformation in the case of a Gaussian action ("free theory"):

$$\big\langle W[\Xi] \big\rangle_{\Xi} = N \int D[\Xi]\ W[\Xi]\ e^{-\frac{\Xi^t M^{-1} \Xi}{2}} = W[\frac{\delta}{i\,\delta \mathcal{J}}] e^{-\frac{\mathcal{J}^t M \mathcal{J}}{2}} \Big|_{\mathcal{J}=0} = e^{-\frac{\frac{\delta}{i\delta \Xi_0} M \frac{\delta}{i\delta \Xi_0}}{2}} W[\Xi_0] \Big|_{\Xi_0=0} \tag{84}$$

for any functional $W[\Xi]$. Here notation has been condensed even further, with all momenta, integration signs and measures suppressed.[7] Legendre machinery is used for the last equality sign. The "classical field," $\Xi_0 = iM\mathcal{J}$, vanishes at $\mathcal{J} = 0$.

The canonical example for $W[\Xi]$ here of course is

$$\big\langle W[\Xi]\big\rangle_\Xi \to \big\langle \mathsf{A}^{ij}_{p\,q}[U,\Xi]\mathsf{A}^{ij}_{p\,q}[U,\Xi]^\dagger\big\rangle_\Xi \ , \tag{85}$$

which, by construction, is the finite N_c generalization of $G^{(\mathrm{real})}_{ab}$.

For this example one immediately realizes that the exponential of the second order differential operator on the right hand side will perform precisely the "sewing" implemented by the Gaussian weight in Eq. (83) as long as M is diagonal in energies as in (82). In this case, the energy ordering ensures that

$$\frac{1}{2}\frac{\delta}{i\delta\Xi_0}M\frac{\delta}{i\delta\Xi_0}\mathsf{A}^{ij}_{p\,q}[U,\Xi_0]\mathsf{A}^{ij}_{p\,q}[U,\Xi]^\dagger = \Big(\frac{\delta}{i\delta\Xi_0}\mathsf{A}^{ij}_{p\,q}[U,\Xi_0]\Big)M\Big(\frac{\delta}{i\delta\Xi_0}\mathsf{A}^{ij}_{p\,q}[U,\Xi]^\dagger\Big) \ . \tag{86}$$

Let me now carry out the derivative with respect to the phase space boundary in analogy with the derivation of the BMS equation sketched in Sec. 2. With $M = M(E)$ a function of the phase space boundary, one immediately finds a general expression for the logarithmic E-derivative of the above expectation value that reads

$$E\partial_E\,\big\langle W[\Xi]\big\rangle_\Xi(E) = -\frac{1}{2}e^{-\frac{\frac{\delta}{i\delta\Xi_0}M(E)\frac{\delta}{i\delta\Xi_0}}{2}}\ \frac{\delta}{i\delta\Xi_0}E\partial_E M(E)\frac{\delta}{i\delta\Xi_0}\ W[\Xi_0]\Big|_{\Xi_0=0} \ . \tag{87}$$

The main point about this seemingly trivial exercise is its use in conjunction with what is already known about functional derivatives of the amplitudes of interest: Eqns. (74), (76). Note that $\partial_E M(E)$ will force the variations in the factor taken down from the exponential to be at the phase space boundary. Choosing $W[\Xi] = \mathsf{A}^{ij}_{p\,q}[U,\Xi]\mathsf{A}^{ij}_{p\,q}[U,\Xi]^\dagger$, the square of the generating functional of $q\bar{q}\,g^n_{\mathrm{soft}}$ amplitudes, this will lead to the appearance of the square of the generating functional of $q\bar{q}g\,g^n_{\mathrm{soft}}$ on the right hand side as follows from (76), (77).

This marks an important difference compared to the BMS case at finite N_c: the equation does not close. Instead higher and higher hard correlators will enter: Clearly, an evolution equation for the $q\bar{q}g\,g^n_{\mathrm{soft}}$ amplitudes will then couple in turn to the corresponding object for $q\bar{q}g^2\,g^n_{\mathrm{soft}}$ amplitudes and so forth. One is faced with an infinite hierarchy. All quantities encountered in this hierarchy can be defined in complete analogy to the examples presented in (68) and (75). One simply uses the desired combination of U-factors as the "bare" term in the tower of amplitudes one is interested in.

This implies that firstly, it is not sufficient to consider only the evolution equation for $q\bar{q}g^n_{\mathrm{soft}}$ amplitudes alone. Instead one has to capture the complete infinite coupled hierarchy of equations in one. Secondly, it indicates that such a step is possible, since the objects of interest have already been identified and indeed can be written down in a general form. The evolution equations then follow from (87).

I therefore proceed to define the (tree level) generating functional based on m hard particles, a general "antenna pattern," by

$$\mathsf{A}^{i_1\ldots i_n}_{p_1\ldots p_m}[U,\Xi]\mathsf{A}^{i_1\ldots i_n}_{p_1\ldots p_m}[V,\Xi]^\dagger = \mathsf{U}[\Xi,U]\ (UV^\dagger)^{(\dagger)}_{p_1}\otimes\ldots\otimes(UV^\dagger)^{(\dagger)}_{p_m}\ \mathsf{U}[\Xi,V]^\dagger\Big|_{U=V} \ . \tag{88}$$

The factor $(UV^\dagger)^{(\dagger)}_{p_1}\otimes\ldots\otimes(UV^\dagger)^{(\dagger)}_{p_m}$ in this expression generalizes the expression for the bare particles of the $q\bar{q}$ case (the $\mathsf{A}^{ij}_{p\,q}[U,\Xi]\mathsf{A}^{ij}_{p\,q}[U,\Xi]^\dagger$ from above) from 2 to m hard legs, and the $\mathsf{U}[\Xi,U]\ \ldots\ \mathsf{U}[\Xi,V]^\dagger\Big|_{U=V}$ implement the showering. I have given only q and $\bar{q}$ factors as any gluon can be written in terms of these as $(U\tilde{V}^\dagger)^{ab}_k = 2\mathrm{tr}(t^a(UV^\dagger)^\dagger_k t^b(UV^\dagger)_k)$.

One can now study the evolution of the expectation value of a given object of the type (88) or directly study a generating functional for *all* such objects in one go. The latter is given by

$$\mathsf{W}[j^\dagger,j] := \Big\langle \mathsf{U}[\Xi,U]\ e^{i2\mathrm{tr}j^\dagger UV^\dagger + i2\mathrm{tr}(UV^\dagger)^\dagger j}\ \mathsf{U}[\Xi,V]^\dagger\Big\rangle_\Xi\Big|_{U=V} \tag{89}$$

[7] For compactness I will do so throughout this section. Where indices and momenta are shown it is with the understanding that there will be integration conventions for repeated momenta with $d\Phi_k$ as the measure.

with integrals in the source exponents understood. Appropriate variations with respect to j and $j^\dagger$ at $j=0$ will then select a given tower of in $\to$ hard $g_{\rm soft}^n$ probabilities entering the hierarchy. Since, besides the phase space constraints contained in M, it is the particle content of the bare terms that define the amplitudes, a notation that emphasizes this is needed. To this end I will write

$$\begin{aligned}\frac{\delta}{i\delta j_{p_1}^{(\dagger)}}\dots\frac{\delta}{i\delta j_{p_m}^{(\dagger)}}\mathsf{W}[j^\dagger,j] =&\Big\langle \mathsf{U}[\Xi,U]\ (UV^\dagger)_{p_1}^{(\dagger)}\otimes\dots\otimes(UV^\dagger)_{p_m}^{(\dagger)}\ \mathsf{U}[\Xi,V]^\dagger\Big\rangle_\Xi\Big|_{U=V}\\ =:&\Big\langle\ (UV^\dagger)_{p_1}^{(\dagger)}\otimes\dots\otimes(UV^\dagger)_{p_m}^{(\dagger)}\ \Big\rangle_{UV^\dagger}^{\rm real}\end{aligned}\tag{90}$$

where the $UV^\dagger$ average in the second line is to be taken with a yet unknown weight that reproduces the l.h.s.. The idea is to interpret the soft gluon average as an average over $UV^\dagger$ configurations. This is completely legal, if somewhat formal as long as the only definition of the weight of the average is through (90).

Nevertheless, it is this reinterpretation that will allow to make contact with the Fokker-Planck formulation of Sec. 3 and that is all that is really needed.

For $m=2$ and an adapted choice for the color structure one encounters

$$\langle{\rm tr}((UV^\dagger)_p^\dagger(UV^\dagger)_q)/N_c\rangle_{UV^\dagger}^{\rm real} =: G_{ab}^{(\rm real)}(E)\tag{91}$$

and thus provides the precise definition of what was still a bit vague in (15). Eq. (90) is of course much more general than that: it defines the real emission part for a general antenna pattern, as these will all be needed for the evolution equations. Inclusion of virtual contributions will be discussed below. Note that the eikonal lines appearing in (90) have the interpretation of a product $UV^\dagger$ of contributions from both the amplitude and the conjugate amplitude.

With (84) one then has

$$\mathsf{W}[j^\dagger,j]=e^{-\frac{\frac{\delta}{i\delta\Xi_0}M(E)\frac{\delta}{i\delta\Xi_0}}{2}}\mathsf{U}[\Xi_0,U]\ e^{i2{\rm tr}j^\dagger UV^\dagger+i2{\rm tr}(UV^\dagger)^\dagger j}\ \mathsf{U}[\Xi_0,V]^\dagger\Big|_{U=V}=\Big\langle e^{i2{\rm tr}j^\dagger UV^\dagger+i2{\rm tr}(UV^\dagger)^\dagger j}\Big\rangle_{UV^\dagger}^{\rm real}\ .\tag{92}$$

This demonstrates that $\mathsf{W}[j^\dagger,j]$ is in complete analogy with (the real emission part of) $\bar{Z}[J,J^\dagger]$ in the derivation of the JIMWLK equation [16].

To find the evolution equation for this (meta-) functional and with it the infinite hierarchy of evolution equations alluded to above, all that is left to do, is to put together Eqns. (87), (86), and (74) to get the dependence on phase space boundaries for arbitrary UV-correlators (bare m-jet probabilities). This yields the r.h.s. for the real emission contribution:

$$\begin{aligned}&e^{-\frac{\frac{\delta}{i\delta\Xi_0}M(E)\frac{\delta}{i\delta\Xi_0}}{2}}\mathsf{U}[\Xi_0,U]\\ &\Big\{g\,J^\mu_{lk}\tilde{U}^{ab}_k i\bar{\nabla}^a_{U_l}\Big\}[E\partial_E M(E)]^{bb'}_{ll'\,\mu\mu'}\Big\{g\,J^{\mu'}_{l'k'}\tilde{V}^{a'b'}_{k'}i\bar{\nabla}^a_{V'_l}\Big\}\ e^{i2{\rm tr}j^\dagger UV^\dagger+i2{\rm tr}(UV^\dagger)^\dagger j}\\ &\mathsf{U}[\Xi_0,V]^\dagger\Big|_{U=V}\ .\end{aligned}\tag{93}$$

To arrive at a meaningful answer one still has to supplement virtual corrections. They are correctly incorporated as usual, by simply subtracting the corresponding term proportional to the original transition probability. This was done in the derivation of the BMS equation and it also applies to the full hierarchy. For concreteness, and without loss of generality, I will demonstrate how to achieve this for the case of m hard particles shown in Eq. (90).

To understand the constraints on the virtual corrections, let me first study the real emission contribution in some more detail. Omitting, for the moment, the operator that implements the soft gluon shower and the expectation value in the expression for the real emission part,

$$e^{-\frac{\frac{\delta}{i\delta\Xi_0}M(E)\frac{\delta}{i\delta\Xi_0}}{2}}\mathsf{U}[\Xi_0,U]\dots\mathsf{U}[\Xi_0,V]^\dagger\Big|_{U=V}\ ,$$

one has to consider

$$\Big\{ g\, J^{\mu}_{lk} \tilde{U}^{ab}_{k} i\bar{\nabla}^{a}_{U_l} \Big\} [E\partial_E M(E)]_{ll'} \delta^{bb'} g_{\mu\mu'} \Big\{ g\, J^{\mu'}_{l'k'} \tilde{V}^{a'b'}_{k'} i\bar{\nabla}^{a}_{V_{l'}} \Big\}\ (UV^\dagger)^{(\dagger)}_{p_1} \otimes \ldots \otimes (UV^\dagger)^{(\dagger)}_{p_m} \tag{94}$$

and evaluate the variations. Inserting the BMS choice (82) for M, it is easy to collect all the factors and to make the k integration explicit (I keep the integration convention for l and l' for readability):

$$\begin{aligned}(94) &= \int \frac{d\omega_k}{\omega_k} E\delta(E-\omega_k) \frac{d\Omega_k}{4\pi} \frac{\alpha_s}{\pi} w_{ll'}(k) (U\tilde{V}^\dagger)^{ab}_k i\bar{\nabla}^a_{U_l} i\bar{\nabla}^b_{V_{l'}}\ (UV^\dagger)^{(\dagger)}_{p_1} \otimes \ldots \otimes (UV^\dagger)^{(\dagger)}_{p_m} \\ &= \int \frac{d\Omega_k}{4\pi} \frac{\alpha_s}{\pi} w_{ll'}(k) (U\tilde{V}^\dagger)^{ab}_k i\bar{\nabla}^a_{U_l} i\bar{\nabla}^b_{V_{l'}}\ (UV^\dagger)^{(\dagger)}_{p_1} \otimes \ldots \otimes (UV^\dagger)^{(\dagger)}_{p_m}\Big|_{\omega_k=E} . \end{aligned} \tag{95}$$

In this expression ω_l and ω'_l will be made hard when the functional derivatives hit the U and V. The energy part of the corresponding phase space integrals will be eliminated at the same time. In summary one will end up with 3 "hard" momenta k, l, l'.

Carrying out the differentiations a sum over pairings will emerge, with different color structures depending if one hits q or $\bar{q}$ lines. Concentrating on the individual terms in this sum, it is obvious that one has to deal with 4 different combinations, corresponding to the pairings qq, $q\bar{q}$, $\bar{q}q$, and $\bar{q}\bar{q}$. This is a structure well known from the the derivation of both the Balitsky hierarchy [4] and the JIMWLK equation in [16]. There I had first to unearth the invariant vector fields from a diagrammatic calculation that has started off from an entirely different setup. Since this tool is already available, I will not go through the line of argument used there, but instead make use of the unifying properties of these operators right away without exploring these structures explicitly.

This may be done by noting that Eq. (94) allows for a very elegant rewrite:

$$(93) = \frac{\alpha_s}{2\pi} w_{pq}(k) u(k) (U\tilde{V}^\dagger)^{ab}_k \big(i\bar{\nabla}^a_{(UV^\dagger)_p} i\nabla^b_{(UV^\dagger)_q} + i\bar{\nabla}^a_{(UV^\dagger)_q} i\nabla^b_{(UV^\dagger)_p} \big)\ (UV^\dagger)^{(\dagger)}_{p_1} \otimes \ldots \otimes (UV^\dagger)^{(\dagger)}_{p_m} . \tag{96}$$

Eq. (96) follows from a direct correspondence of $i\bar{\nabla}^a_{(UV^\dagger)_k}$ with $i\bar{\nabla}^a_{U_k}$ acting on the amplitude and $i\nabla^a_{(UV^\dagger)_k}$ with $i\bar{\nabla}^a_{V_k}$ acting on the complex conjugate factor. In Eq. (96) it is possible to anticipate that the energies will all be made hard by functional differentiation on the hard factors $(UV^\dagger)_{p_i}$ and thus to go back to the original definition of the variations without the energy δ-function. This allows to use an integration convention that employs the solid angle integrations only and a symmetric treatment of k, p, q in this equation. Reinstating the shower-operators and the expectation value, one arrives at the real emission part of the evolution equation

$$\Big\langle \frac{\alpha_s}{2\pi} w_{pq}(k) u(k) (U\tilde{V}^\dagger)^{ab}_k \big(i\bar{\nabla}^a_{(UV^\dagger)_p} i\nabla^b_{(UV^\dagger)_q} + i\bar{\nabla}^a_{(UV^\dagger)_q} i\nabla^b_{(UV^\dagger)_p} \big)\ (UV^\dagger)^{(\dagger)}_{p_1} \otimes \ldots \otimes (UV^\dagger)^{(\dagger)}_{p_m} \Big\rangle_{UV^\dagger}(E) . \tag{97}$$

To discuss virtual corrections one needs to consider the physical process at hand. While the real emissions are confined to the *inside* regions and hence carry a factor $u(k)$, virtual corrections appear everywhere and therefore have no such factor. Moreover the factor $(\tilde{U}V^\dagger)^{ab}_k$ will be replaced by δ^{ab} and and the vector fields will have to act twice *within* either the amplitude or its complex conjugate, leading to $i\nabla^a_{(UV^\dagger)_p} i\nabla^a_{(UV^\dagger)_q}$ and an analogous barred contribution instead of the mixed ones in Eq. (97). For symmetry reasons one therefore expects the virtual corrections to read

$$\Big\langle \frac{\alpha_s}{2\pi} w_{pq}(k) \big(i\nabla^a_{(UV^\dagger)_p} i\nabla^a_{(UV^\dagger)_q} + i\bar{\nabla}^a_{(UV^\dagger)_q} i\bar{\nabla}^a_{(UV^\dagger)_p} \big)\ (UV^\dagger)^{(\dagger)}_{p_1} \otimes \ldots \otimes (UV^\dagger)^{(\dagger)}_{p_m} \Big\rangle_{UV^\dagger}(E) \tag{98}$$

where the overall sign and normalization is fixed by real virtual cancellation: only with this choice do they add up to an infrared finite Fokker-Planck Hamiltonian, which happens to coincide with the expression conjectured in Sec. 2. Indeed, the operator appearing in the sum of (97) and (98) is nothing but $-H_{\rm ng}$ of Eq. (34a) with the $f^{(i)}$ completely determined. In particular, there is no room for any N_c dependence of the coefficients from this argument.

To summarize, it has been shown that

$$E\partial_E \left\langle e^{i2\mathrm{tr} j^\dagger UV^\dagger + i2\mathrm{tr}(UV^\dagger)^\dagger j} \right\rangle_{UV^\dagger}(E) = -\Big\langle\ H_{\rm ng}\ e^{i2\mathrm{tr} j^\dagger UV^\dagger + i2\mathrm{tr}(UV^\dagger)^\dagger j}\ \Big\rangle_{UV^\dagger}(E) . \tag{99}$$

The only argument still missing is about how to reverse the step leading from (18) to (21).

This follows from the fact that the identity holds for arbitrary j and $j^\dagger$, i.e. the fact that the above is completely independent of the type of correlator considered. The result is, as advertised in Sec. 3, the evolution equation

$$E\partial_E\ \hat{Z}_E[UV^\dagger] = -H_{\rm ng}\ \hat{Z}_E[UV^\dagger] \tag{100}$$

for the weight $Z_E[UV^\dagger]$ used to define the averages $\langle\ldots\rangle_{UV^\dagger}(E)$. This completes the derivation of Eq. (36).

I will close this section with the explicit expressions for the non-factorized version of the BMS equation as it emerges from this discussion, to point out how the N_c limit comes about and to illustrate once more the real-virtual cancellations with a concrete example. As this is concerned with emissions from a bare $q\bar{q}$ jet, one needs the operator and probabilities

$$\hat{G}_{pq} :=\mathrm{tr}((UV^\dagger)^\dagger_p(UV^\dagger)_q)/N_c \tag{101a}$$

$$G_{pq}(E) :=\langle\hat{G}_{pq}\rangle_{UV^\dagger}(E) = \frac{1}{N_c}\frac{\delta}{i\delta(j_p)_{ij}}\frac{\delta}{i\delta(j_p)_{ij}}\mathsf{W}_E[j^\dagger,j]\ . \tag{101b}$$

The resulting equation then is

$$\begin{aligned}&E\partial_E\langle\mathrm{tr}((UV^\dagger)^\dagger_p(UV^\dagger)_q)/N_c\rangle_{UV^\dagger}(E)\\ &\quad=\int\frac{d\Omega_k}{4\pi}\frac{\alpha_s}{\pi}w_{pq}(k)\ \Big\langle u(k)(\tilde{UV^\dagger})^{ab}_k\ 2\frac{\mathrm{tr}(t^a(UV^\dagger)_pt^b(UV^\dagger)_q)}{N_c}-2C_{\rm f}\frac{\mathrm{tr}((UV^\dagger)^\dagger_p(UV^\dagger)_q)}{N_c}\Big\rangle_{UV^\dagger}(E)\ .\end{aligned} \tag{102}$$

The first term on the r.h.s., carrying the factor $u(k)$, originates from the real contribution, the second from the virtual one. By (29) this is equivalent to

$$E\partial_EG_{pq}(E) = \int\frac{d\Omega_k}{4\pi}\frac{\alpha_s}{\pi}w_{pq}(k)\Big\langle u(k)\big(\hat{G}_{pk}\hat{G}_{kq}N_c-\frac{\hat{G}_{pq}}{N_c}\big)-2C_{\rm f}\hat{G}_{pq}\Big\rangle(E) \tag{103a}$$

$$=\int\frac{d\Omega_k}{4\pi}\bar{\alpha}_sw_{pq}(k)\Big\langle u(k)\big(\hat{G}_{pk}\hat{G}_{kq}-\hat{G}_{pq}\big)-\frac{2C_{\rm f}}{N_c}(1-u(k))\hat{G}_{pq}\Big\rangle(E)\ , \tag{103b}$$

which in the large N_c limit factorizes into the analogues of the BK that triggered these explorations, the BMS equation Eqns. (3) and (4) respectively. To see this explicitly, use (84) to represent the averaging involved in

$$G_{pq}(E) = \langle\hat{G}_{pq}\rangle_{UV^\dagger}(E) = \langle\mathsf{A}^{ij}_{p\,q}[U,\Xi]\mathsf{A}^{ij}_{p\,q}[U,\Xi]^\dagger\rangle_\Xi \tag{104}$$

and then repeat the argument of [21], Eq. (50). It is worth emphasizing that the real-virtual cancellation argument of BMS restated in Sec. 2 applies already to Eq. (103) itself, without any reference to the factorization argument. Infrared finiteness in the Sudakov term follows directly from the argument of BMS while in the other term it is the definition of $\hat{G}$ that saves the day. This provides an illustration of how the more general reasoning above will lead to infrared finite results via real-virtual cancellation in *any* evolution equation for color singlet objects contained in the new evolution equation.

8 Conclusions

The main result of this paper is the evolution equation (36) that generalizes the BMS equation to finite N_c. The result is functional and represents an infinite hierarchy of coupled equations just as its small x analogue, the JIMWLK equation. As in this case, a Langevin formulation has been derived that hopefully can be implemented efficiently as a numerical simulation.

More subtle benefits lie in the method used to prove the result. This was based on a formulation of strongly ordered soft emission amplitudes using shower operators that can be used to generate soft gluon clouds around any hard seed. The ordered nature of these is the main calculational benefit that eventually allowed the derivation of the new evolution equation. This concept should prove useful in most of the new applications briefly touched upon below.

The amplitudes were then translated into probabilities and several forms for taking the expectation values were given. Each of these has its own advantage as shown by their use in the derivation of the evolution equation. This in the end has allowed to derive this equation by simply differentiating the expressions found with surprisingly simple manipulations. The simplicity of these steps allows the modification of the equation to applications with other global and non-global jet observables.

By the close analogy with JIMWLK and BK, some of the results obtained in the asymptotic regime, such as the presence of the saturation scale clearly seem to map on universal features of the jet evolution, such as the existence of $\theta_{\rm crit}$ discussed by BMS. Further explorations along these lines appear to be promising.

Conversely, it should be possible to use the strategy employed here to rederive the JIMWLK equation. In this case, quantities that are less inclusive than the dipole cross section in DIS should become accessible. Diffraction or the formulation of observables that focus in on small impact parameters would emerge by formulating appropriate phase space constraints. The definition of observables that are infrared safe regardless of impact parameter and gluon density encountered is another item on this list. Such a step will certainly come at the price of an additional term in the evolution equation just like the Sudakov term in BMS, but would put applications of that equation on solid ground even for small targets like the proton. This should allow a new level of insight for the application of saturation ideas to HERA data and ultimately also help with applications to RHIC and LHC physics.

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A Color algebra of the induction step

The color algebra pattern can be best understood by looking at the small values for n. This uncovers the pattern for the general case. Notation below is simplified in order to expose the structure of terms:

- n=0: starting from $U_p^\dagger U_q$ one arrives at

$$-J_{pa}U_p^\dagger t^a U_q + J_{aq}U_p^\dagger t^a U_q = U_p^\dagger t^a U_q A_{pq}(k)$$

- n=1: starting from $(U_p^\dagger t^{a_1} U_q)\tilde{U}^{a_1 b_1}$ one finds

$$\begin{aligned}-J_{pa}U_p^\dagger t^a t^{a_1} U_q \tilde{U}^{a_1 b_1} + J_{a_1 a}(U_p^\dagger [t^a, t^{a_1}] U_q)\tilde{U}^{a_1 b_1} + J_{qa}U_p^\dagger t^{a_1} t^a U_q)\tilde{U}^{a_1 b_1}\\ = (J_{a_1 a} - J_{pa})U_p^\dagger t^a t^{a_1} U_q U^{a_1 b_1} + (J_{qa} - J_{a_1 a})U_p^\dagger t^{a_1} t^a U_q \tilde{U}^{a_1 b_1}\end{aligned}$$

- n=2: starting from $(U_p^\dagger t^{a_1} t^{a_2} U_q)\tilde{U}^{a_1 b_1}\tilde{U}^{a_2 b_2}$ one obtains

$$\begin{aligned}&- J_{pa}(U_p^\dagger t^a t^{a_1} t^{a_2} U_q \tilde{U}^{a_1 b_1}\\ &+ J_{a_1 a}(U_p^\dagger [t^a, t^{a_1}] t^{a_2} U_q)\tilde{U}^{a_1 b_1} + J_{a_2 a}(U_p^\dagger t^{a_1} [t^a, t^{a_2}] U_q)\tilde{U}^{a_1 b_1}\\ &+ J_{qa}(U_p^\dagger t^{a_1} t^{a_2} t^a U_q))\tilde{U}^{a_1 b_1}\end{aligned}$$

The general step is now obvious: to collect terms with the same color structure, the first term (from the $U^\dagger$) pairs with the first term of the first commutator, then iteratively the second term of one commutator with the first term of the next. This continues all the way through until the last remaining term of the last commutator pairs up with the term of the U factor. In all the pairings, the signs will alternate, yielding the $A_{ll'}(k)$. Setting the labels accordingly then leads to the induction step given in Eq. (71).

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